Linear-Phase FIR Transfer Functions

- It is nearly impossible to design a linear-phase IIR transfer function
- It is always possible to design an FIR transfer function with an exact linear-phase response
- Consider a causal FIR transfer function $H(z)$ of length $N+1$, i.e., of order N:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

Linear-Phase FIR Transfer Functions

- The above transfer function has a linear phase, if its impulse response $h[n]$ is either **symmetric**, i.e.,

$$h[n] = h[N-n], \;\; 0 \le n \le N$$

or is **antisymmetric**, i.e.,

$$h[n] = -h[N-n], \;\; 0 \le n \le N$$

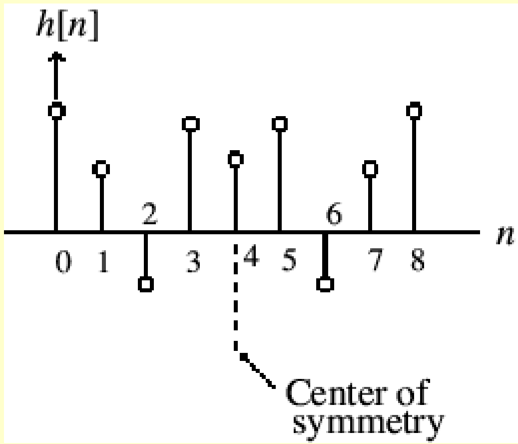
Linear-Phase FIR Transfer Functions

- Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions
- For an antisymmetric FIR filter of odd length, i.e., N even

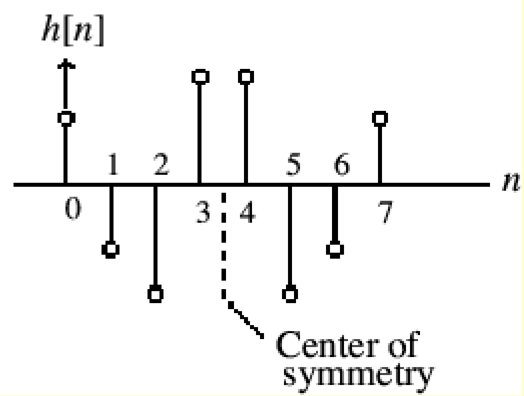
$$h[N/2] = 0$$

- We examine next the each of the 4 cases

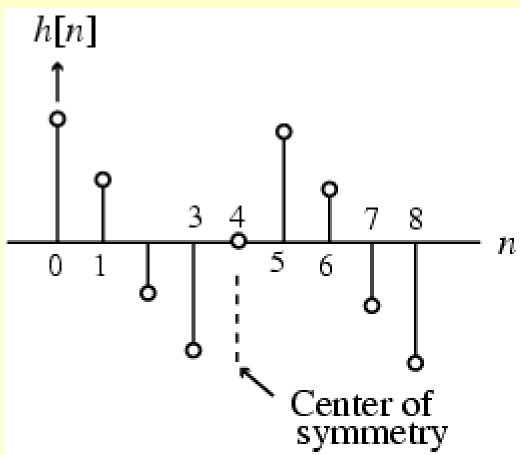
Linear-Phase FIR Transfer Functions



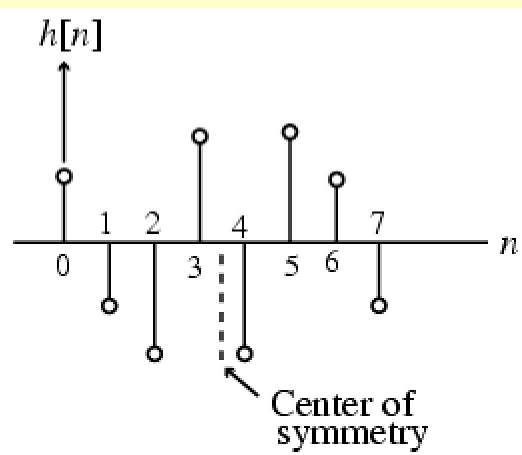
Type 1: $N = 8$



Type 2: $N = 7$



Type 3: $N = 8$



Type 4: $N = 7$

Linear-Phase FIR Transfer Functions

Type 1: Symmetric Impulse Response with Odd Length

- In this case, the degree N is even
- Assume $N = 8$ for simplicity
- The transfer function $H(z)$ is given by

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

Linear-Phase FIR Transfer Functions

- Because of symmetry, we have $h[0] = h[8]$, $h[1] = h[7]$, $h[2] = h[6]$, and $h[3] = h[5]$
- Thus, we can write

$$H(z) = h[0](1 + z^{-8}) + h[1](z^{-1} + z^{-7})$$
$$+ h[2](z^{-2} + z^{-6}) + h[3](z^{-3} + z^{-5}) + h[4]z^{-4}$$
$$= z^{-4}\{h[0](z^{4} + z^{-4}) + h[1](z^{3} + z^{-3})$$
$$+ h[2](z^{2} + z^{-2}) + h[3](z + z^{-1}) + h[4]\}$$

Linear-Phase FIR Transfer Functions

- The corresponding frequency response is then given by

$$H(e^{j\omega}) = e^{-j4\omega}\{2h[0]\cos(4\omega) + 2h[1]\cos(3\omega)$$
$$+ 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]\}$$

- The quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$

Linear-Phase FIR Transfer Functions

- The phase function here is given by

$$\theta(\omega) = -4\omega + \beta$$

where β is either 0 or π, and hence, it is a linear function of ω in the generalized sense

- The group delay is given by

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4$$

indicating a constant group delay of 4 samples

Linear-Phase FIR Transfer Functions

- In the general case for Type 1 FIR filters, the frequency response is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2}\tilde{H}(\omega)$$

where the **amplitude response** $\tilde{H}(\omega)$, also called the **zero-phase response**, is of the form

$$\tilde{H}(\omega) = h[\tfrac{N}{2}] + 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\cos(\omega n)$$

Linear-Phase FIR Transfer Functions

- Example - Consider

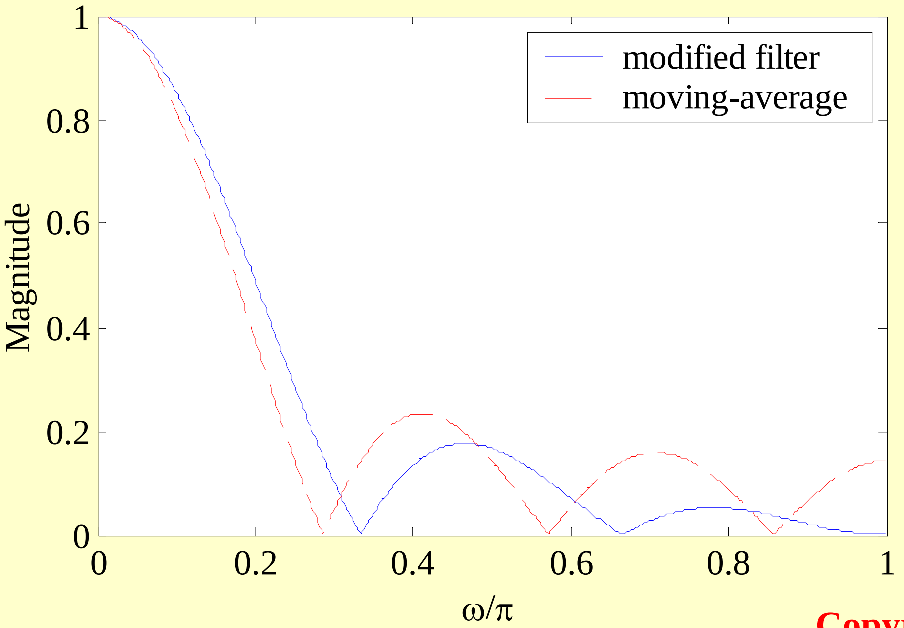
$$H_0(z) = \frac{1}{6}[\frac{1}{2} + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \frac{1}{2} z^{-6}]$$

which is seen to be a slightly modified version of a length-7 moving-average FIR filter

- The above transfer function has a symmetric impulse response and therefore a linear phase response

Linear-Phase FIR Transfer Functions

- A plot of the magnitude response of $H_0(z)$ along with that of the 7-point moving-average filter is shown below



Linear-Phase FIR Transfer Functions

- Note the improved magnitude response obtained by simply changing the first and the last impulse response coefficients of a moving-average (MA) filter
- It can be shown that we an express

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) \cdot \tfrac{1}{6}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

which is seen to be a cascade of a 2-point MA filter with a 6-point MA filter

- Thus, $H_0(z)$ has a double zero at $z = -1$, i.e., $(\omega = \pi)$

Linear-Phase FIR Transfer Functions

Type 2: Symmetric Impulse Response with Even Length

- In this case, the degree N is odd
- Assume $N = 7$ for simplicity
- The transfer function is of the form

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7}$$

Linear-Phase FIR Transfer Functions

- Making use of the symmetry of the impulse response coefficients, the transfer function can be written as

$$H(z) = h[0](1 + z^{-7}) + h[1](z^{-1} + z^{-6}) + h[2](z^{-2} + z^{-5}) + h[3](z^{-3} + z^{-4})$$

$$= z^{-7/2}\{h[0](z^{7/2} + z^{-7/2}) + h[1](z^{5/2} + z^{-5/2}) + h[2](z^{3/2} + z^{-3/2}) + h[3](z^{1/2} + z^{-1/2})\}$$

Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j7\omega/2}\{2h[0]\cos(\tfrac{7\omega}{2}) + 2h[1]\cos(\tfrac{5\omega}{2})$$
$$+ 2h[2]\cos(\tfrac{3\omega}{2}) + 2h[3]\cos(\tfrac{\omega}{2})\}$$

- As before, the quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$

Linear-Phase FIR Transfer Functions

- Here the phase function is given by

$$\theta(\omega) = -\frac{7}{2}\omega + \beta$$

where again β is either 0 or π

- As a result, the phase is also a linear function of ω in the generalized sense
- The corresponding group delay is

$$\tau(\omega) = \frac{7}{2}$$

indicating a group delay of $\frac{7}{2}$ samples

Linear-Phase FIR Transfer Functions

- The expression for the frequency response in the general case for Type 2 FIR filters is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2}\tilde{H}(\omega)$$

where the amplitude response is given by

$$\tilde{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n]\cos(\omega(n - \tfrac{1}{2}))$$

Linear-Phase FIR Transfer Functions

Type 3: Antiymmetric Impulse Response with Odd Length

- In this case, the degree N is even
- Assume $N = 8$ for simplicity
- Applying the symmetry condition we get

$$H(z) = z^{-4}\{h[0](z^4 - z^{-4}) + h[1](z^3 - z^{-3}) + h[2](z^2 - z^{-2}) + h[3](z - z^{-1})\}$$

Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j4\omega}e^{-j\pi/2}\{2h[0]\sin(4\omega) + 2h[1]\sin(3\omega) + 2h[2]\sin(2\omega) + 2h[3]\sin(\omega)\}$$

- It also exhibits a generalized phase response given by

$$\theta(\omega) = -4\omega + \frac{\pi}{2} + \beta$$

where β is either 0 or π

Linear-Phase FIR Transfer Functions

- The group delay here is

$$\tau(\omega) = 4$$

indicating a constant group delay of 4 samples

- In the general case

$$H(e^{j\omega}) = je^{-jN\omega/2}\tilde{H}(\omega)$$

where the amplitude response is of the form

$$\tilde{H}(\omega) = 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\sin(\omega n)$$

Linear-Phase FIR Transfer Functions

Type 4: Antiymmetric Impulse Response with Even Length

- In this case, the degree N is even
- Assume $N = 7$ for simplicity
- Applying the symmetry condition we get

$$H(z) = z^{-7/2}\{h[0](z^{7/2} - z^{-7/2}) + h[1](z^{5/2} - z^{-5/2}) + h[2](z^{3/2} - z^{-3/2}) + h[3](z^{1/2} - z^{-1/2})\}$$

Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j7\omega/2} e^{-j\pi/2}\{2h[0]\sin(\tfrac{7\omega}{2}) + 2h[1]\sin(\tfrac{5\omega}{2})$$
$$+ 2h[2]\sin(\tfrac{3\omega}{2}) + 2h[3]\sin(\tfrac{\omega}{2})\}$$

- It again exhibits a generalized phase response given by

$$\theta(\omega) = -\tfrac{7}{2}\omega + \tfrac{\pi}{2} + \beta$$

where β is either 0 or π

Linear-Phase FIR Transfer Functions

- The group delay is constant and is given by

$$\tau(\omega) = \tfrac{7}{2}$$

- In the general case we have

$$H(e^{j\omega}) = j e^{-jN\omega/2} \tilde{H}(\omega)$$

where now the amplitude response is of the form

$$\tilde{H}(\omega) = 2 \sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n] \sin(\omega(n - \tfrac{1}{2}))$$

Linear-Phase FIR Transfer Functions

General Form of Frequency Response

- In each of the four types of linear-phase FIR filters, the frequency response is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\beta} \tilde{H}(\omega)$$

- The amplitude response $\tilde{H}(\omega)$ for each of the four types of linear-phase FIR filters can become negative over certain frequency ranges, typically in the stopband

Linear-Phase FIR Transfer Functions

- The magnitude and phase responses of the linear-phase FIR are given by

$$|H(e^{j\omega})| = |\tilde{H}(\omega)|$$

$$\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta, & \text{for } \tilde{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi, & \text{for } \tilde{H}(\omega) < 0 \end{cases}$$

- The group delay in each case is

$$\tau(\omega) = \frac{N}{2}$$

Linear-Phase FIR Transfer Functions

- Note that, even though the group delay is constant, since in general $|H(e^{j\omega})|$ is not a constant, the output waveform is not a replica of the input waveform
- An FIR filter with a frequency response that is a real function of ω is often called a **zero-phase filter**
- Such a filter must have a noncausal impulse response

Zero Locations of Linear-Phase FIR Transfer Functions

- Consider first an FIR filter with a symmetric impulse response: $h[n] = h[N-n]$
- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = \sum_{n=0}^{N} h[N-n] z^{-n}$$

- By making a change of variable $m = N - n$, we can write

$$\sum_{n=0}^{N} h[N-n] z^{-n} = \sum_{m=0}^{N} h[m] z^{-N+m} = z^{-N} \sum_{m=0}^{N} h[m] z^{m}$$

Zero Locations of Linear-Phase FIR Transfer Functions

- But,

$$\sum_{m=0}^{N} h[m] z^{m} = H(z^{-1})$$

- Hence for an FIR filter with a symmetric impulse response of length $N+1$ we have

$$H(z) = z^{-N} H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **mirror-image polynomial** (MIP)

Zero Locations of Linear-Phase FIR Transfer Functions

- Now consider first an FIR filter with an antisymmetric impulse response:

$$h[n] = -h[N-n]$$

- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = -\sum_{n=0}^{N} h[N-n] z^{-n}$$

- By making a change of variable $m = N - n$, we get

$$-\sum_{n=0}^{N} h[N-n] z^{-n} = -\sum_{m=0}^{N} h[m] z^{-N+m} = -z^{-N} H(z^{-1})$$

Zero Locations of Linear-Phase FIR Transfer Functions

- Hence, the transfer function $H(z)$ of an FIR filter with an antisymmetric impulse response satisfies the condition

$$H(z) = -z^{-N} H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **antimirror-image polynomial** (AIP)

Zero Locations of Linear-Phase FIR Transfer Functions

- It follows from the relation $H(z) = \pm z^{-N} H(z^{-1})$ that if $z = \xi_o$ is a zero of $H(z)$, so is $z = 1/\xi_o$
- Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs
- Hence, a zero at $z = \xi_o$ is associated with a zero at $z = \xi_o^*$

Zero Locations of Linear-Phase FIR Transfer Functions

- Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by

$$z = re^{\pm j\phi}, \quad z = \frac{1}{r}e^{\pm j\phi}$$

- A zero on the unit circle appear as a pair

$$z = e^{\pm j\phi}$$

as its reciprocal is also its complex conjugate

Zero Locations of Linear-Phase FIR Transfer Functions

- Since a zero at $z = \pm 1$ is its own reciprocal, it can appear only singly
- Now a Type 2 FIR filter satisfies

$$H(z) = z^{-N} H(z^{-1})$$

with degree N odd

- Hence $H(-1) = (-1)^{-N} H(-1) = -H(-1)$

implying $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$

Zero Locations of Linear-Phase FIR Transfer Functions

- Likewise, a Type 3 or 4 FIR filter satisfies

$$H(z) = -z^{-N} H(z^{-1})$$

- Thus $H(1) = -(1)^{-N} H(1) = -H(1)$

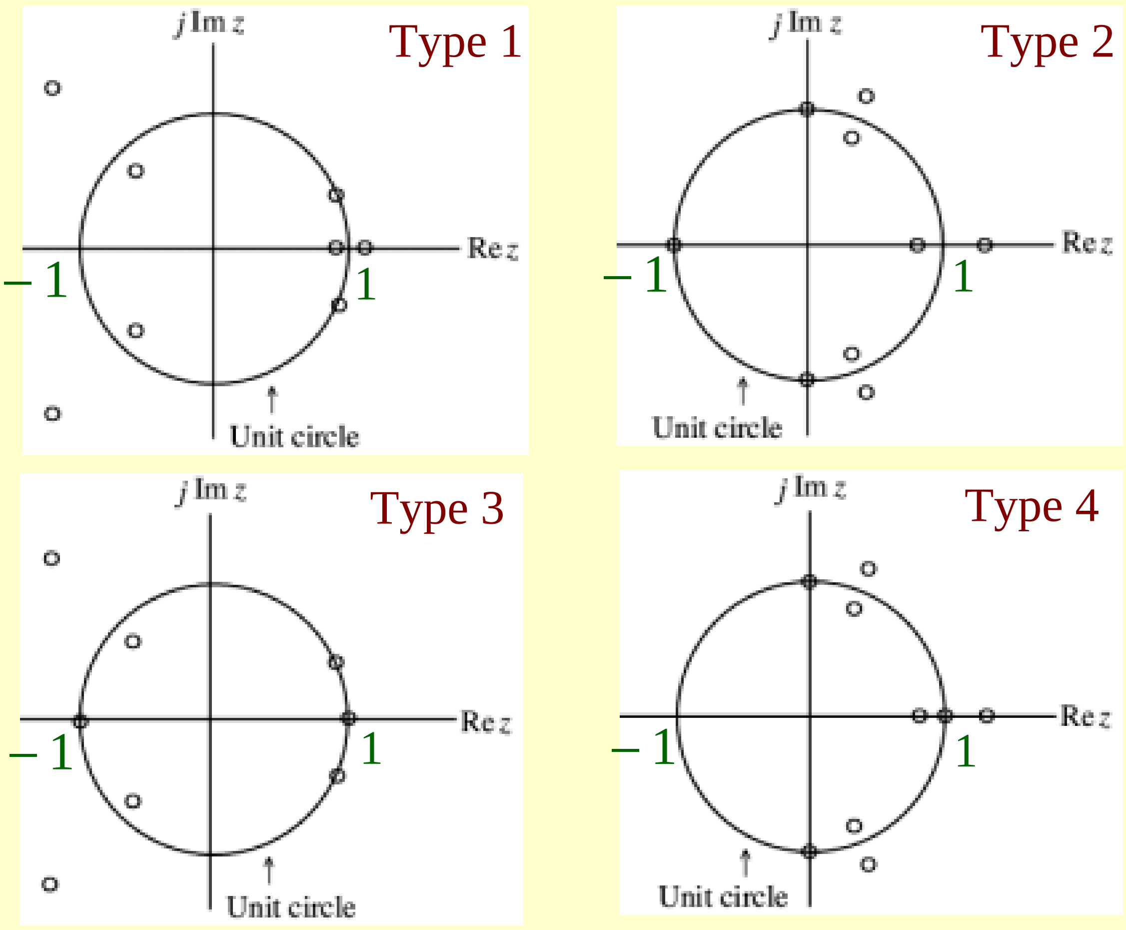
  implying that $H(z)$ must have a zero at $z = 1$

- On the other hand, only the Type 3 FIR filter is restricted to have a zero at $z = -1$ since here the degree N is even and hence,

$$H(-1) = -(-1)^{-N} H(-1) = -H(-1)$$

Zero Locations of Linear-Phase FIR Transfer Functions

- Typical zero locations shown below



Zero Locations of Linear-Phase FIR Transfer Functions

- Summarizing

(1) Type 1 FIR filter: Either an even number or no zeros at $z = 1$ and $z = -1$

(2) Type 2 FIR filter: Either an even number or no zeros at $z = 1$, and an odd number of zeros at $z = -1$

(3) Type 3 FIR filter: An odd number of zeros at $z = 1$ and $z = -1$

Zero Locations of Linear-Phase FIR Transfer Functions

(4) Type 4 FIR filter: An odd number of zeros at $z = 1$, and either an even number or no zeros at $z = -1$

- The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters

Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero $z = -1$
- A Type 3 FIR filter has zeros at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter

Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at $z = 1$
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter

Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a **bounded real** (BR) **transfer function** if

$$|H(e^{j\omega})| \leq 1 \quad \text{for all values of } \omega$$

- Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs

Bounded Real Transfer Functions

- Then the condition $|H(e^{j\omega})| \leq 1$ implies that

$$\left|Y(e^{j\omega})\right|^2 \leq \left|X(e^{j\omega})\right|^2$$

- Integrating the above from $-\pi$ to π, and applying Parseval's relation we get

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \leq \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a **passive structure**
- If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a **lossless system**

Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a **lossless bounded real (LBR) transfer function**
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity