Spectral Transformations of IIR Digital Filters

- Objective - Transform a given lowpass digital transfer function $G_L(z)$ to another digital transfer function $G_D(\hat{z})$ that could be a lowpass, highpass, bandpass or bandstop filter
- z^{-1} has been used to denote the unit delay in the prototype lowpass filter $G_L(z)$ and $\hat{z}^{-1}$ to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion

Spectral Transformations of IIR Digital Filters

- Unit circles in z- and $\hat{z}$ -planes defined by

$$z = e^{j\omega}, \quad \hat{z} = e^{j\hat{\omega}}$$

- Transformation from z-domain to $\hat{z}$-domain given by

$$z = F(\hat{z})$$

- Then

$$G_D(\hat{z}) = G_L\{F(\hat{z})\}$$

Spectral Transformations of IIR Digital Filters

- From $z = F(\hat{z})$, thus $|z| = |F(\hat{z})|$, hence

$$|F(\hat{z})| \begin{cases} >1, & \text{if } |z| > 1 \\ =1, & \text{if } |z| = 1 \\ <1, & \text{if } |z| < 1 \end{cases}$$

- Recall that a stable allpass function $A(z)$ satisfies the condition

Spectral Transformations of IIR Digital Filters

$$\left|A(z)\right| \begin{cases} <1, & \text{if } \left|z\right| > 1 \\ =1, & \text{if } \left|z\right| = 1 \\ >1, & \text{if } \left|z\right| < 1 \end{cases}$$

- Therefore $1/F(\hat{z})$ must be a stable allpass function whose general form is

$$\frac{1}{F(\hat{z})} = \pm \prod_{\ell=1}^{L} \left(\frac{1 - \alpha_{\ell}^{*} \hat{z}}{\hat{z} - \alpha_{\ell}} \right), \quad \left|\alpha_{\ell}\right| < 1$$

Lowpass-to-Lowpass Spectral Transformation

- To transform a lowpass filter $G_L(z)$ with a cutoff frequency ω_c to another lowpass filter $G_D(\hat{z})$ with a cutoff frequency $\hat{\omega}_c$, the transformation is

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1 - \alpha\,\hat{z}}{\hat{z} - \alpha}$$

where α is a function of the two specified cutoff frequencies

Lowpass-to-Lowpass Spectral Transformation

- On the unit circle we have

$$e^{-j\omega} = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}}$$

- From the above we get

$$e^{-j\omega} \mp 1 = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}} \mp 1 = (1 \pm \alpha) \cdot \frac{e^{-j\hat{\omega}} \mp 1}{1 - \alpha e^{-j\hat{\omega}}}$$

- Taking the ratios of the above two expressions

$$\tan(\omega/2) = \left(\frac{1+\alpha}{1-\alpha}\right) \tan(\hat{\omega}/2)$$

Lowpass-to-Lowpass Spectral Transformation

- Solving we get

$$\alpha = \frac{\sin\left((\omega_c - \hat{\omega}_c)/2\right)}{\sin\left((\omega_c + \hat{\omega}_c)/2\right)}$$

- Example - Consider the lowpass digital filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

which has a passband from dc to 0.25π with a 0.5 dB ripple

- Redesign the above filter to move the passband edge to 0.35π

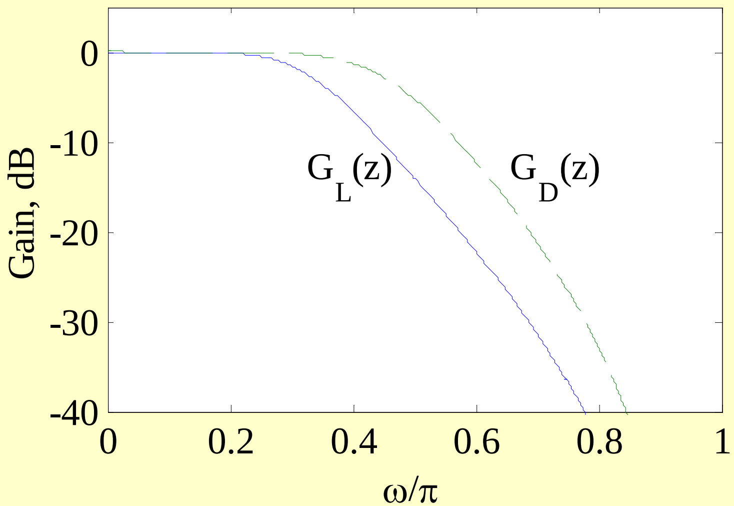
Lowpass-to-Lowpass Spectral Transformation

- Here

$$\alpha = -\frac{\sin(0.05\pi)}{\sin(0.3\pi)} = -0.1934$$

- Hence, the desired lowpass transfer function is

$$G_D(\hat{z}) = G_L(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1} + 0.1934}{1 + 0.1934\,\hat{z}^{-1}}}$$



Lowpass-to-Lowpass Spectral Transformation

- The lowpass-to-lowpass transformation

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1-\alpha\,\hat{z}}{\hat{z}-\alpha}$$

can also be used as highpass-to-highpass, bandpass-to-bandpass and bandstop-to-bandstop transformations

Lowpass-to-Highpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha \hat{z}^{-1}}$$

- The transformation parameter α is given by

$$\alpha = -\frac{\cos((\omega_c + \hat{\omega}_c)/2)}{\cos((\omega_c - \hat{\omega}_c)/2)}$$

where ω_c is the cutoff frequency of the lowpass filter and $\hat{\omega}_c$ is the cutoff frequency of the desired highpass filter

Lowpass-to-Highpass Spectral Transformation

- Example - Transform the lowpass filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

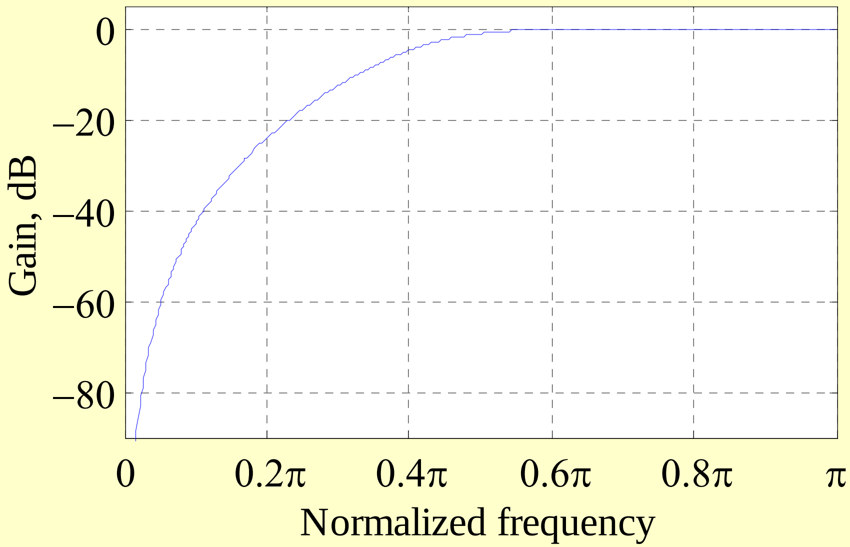
- with a passband edge at 0.25π to a highpass filter with a passband edge at 0.55π
- Here $\alpha = -\cos(0.4\pi)/\cos(0.15\pi) = -0.3468$
- The desired transformation is

$$z^{-1} = -\frac{\hat{z}^{-1} - 0.3468}{1 - 0.3468\hat{z}^{-1}}$$

Lowpass-to-Highpass Spectral Transformation

- The desired highpass filter is

$$G_D(\hat{z}) = G(z)\Big|_{z^{-1} = -\frac{\hat{z}^{-1} - 0.3468}{1 - 0.3468\,\hat{z}^{-1}}}$$



Lowpass-to-Highpass Spectral Transformation

- The lowpass-to-highpass transformation can also be used to transform a highpass filter with a cutoff at ω_c to a lowpass filter with a cutoff at $\hat{\omega}_c$

  and transform a bandpass filter with a center frequency at ω_o to a bandstop filter with a center frequency at $\hat{\omega}_o$

Lowpass-to-Bandpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + \frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + 1}$$

Lowpass-to-Bandpass Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos\big((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\big)}{\cos\big((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\big)}$$

$$\beta = \cot\big((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\big)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandpass filter

Lowpass-to-Bandpass Spectral Transformation

- Special Case - The transformation can be simplified if $\omega_c = \hat{\omega}_{c2} - \hat{\omega}_{c1}$
- Then the transformation reduces to

$$z^{-1} = -\hat{z}^{-1} \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\, \hat{z}^{-1}}$$

where $\alpha = \cos \hat{\omega}_o$ with $\hat{\omega}_o$ denoting the desired center frequency of the bandpass filter

Lowpass-to-Bandstop Spectral Transformation

- Desired transformation

$$z^{-1} = \frac{\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + \frac{1-\beta}{1+\beta}}{\frac{1-\beta}{1+\beta}\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + 1}$$

Lowpass-to-Bandstop Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos\left((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\right)}{\cos\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)}$$

$$\beta = \tan\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandstop filter

Least Integral-Squared Error Design of FIR Filters

- Let $H_d(e^{j\omega})$ denote the desired frequency response
- Since $H_d(e^{j\omega})$ is a periodic function of ω with a period 2π, it can be expressed as a Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

where

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \le n \le \infty$$

Least Integral-Squared Error Design of FIR Filters

- In general, $H_d(e^{j\omega})$ is piecewise constant with sharp transitions between bands
- In which case, $\{h_d[n]\}$ is of infinite length and noncausal
- Objective - Find a finite-duration $\{h_t[n]\}$ of length $2M+1$ whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$ in some sense

Least Integral-Squared Error Design of FIR Filters

- Commonly used approximation criterion - Minimize the integral-squared error

$$\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_t(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 d\omega$$

where

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n}$$

Least Integral-Squared Error Design of FIR Filters

- Using Parseval's relation we can write

$$\Phi = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

$$= \sum_{n=-M}^{M} \left| h_t[n] - h_d[n] \right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

- It follows from the above that Φ is minimum when $h_t[n] = h_d[n]$ for $-M \leq n \leq M$
- $\Rightarrow$ Best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncation

Least Integral-Squared Error Design of FIR Filters

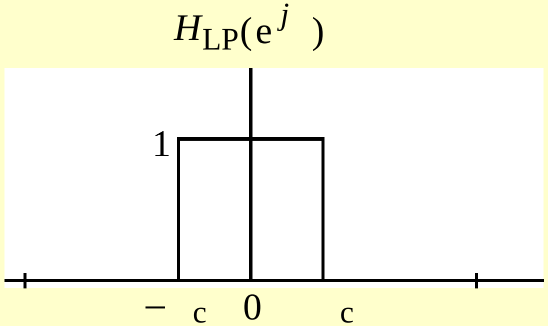
- A causal FIR filter with an impulse response $h[n]$ can be derived from $h_t[n]$ by delaying:

$$h[n] = h_t[n - M]$$

- The causal FIR filter $h[n]$ has the same magnitude response as $h_t[n]$ and its phase response has a linear phase shift of ωM radians with respect to that of $h_t[n]$

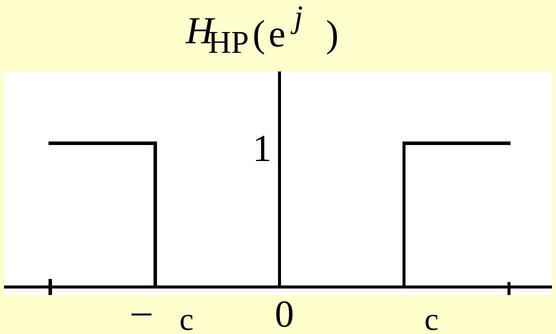
Impulse Responses of Ideal Filters

- Ideal lowpass filter -



$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$

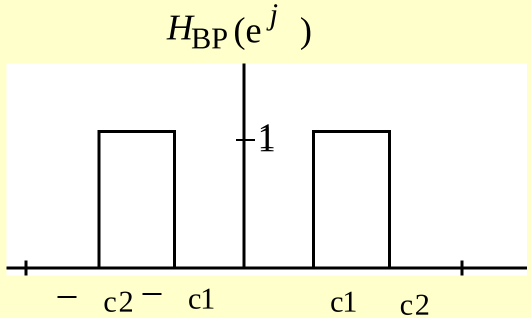
- Ideal highpass filter -



$$h_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n}, & n \ne 0 \end{cases}$$

Impulse Responses of Ideal Filters

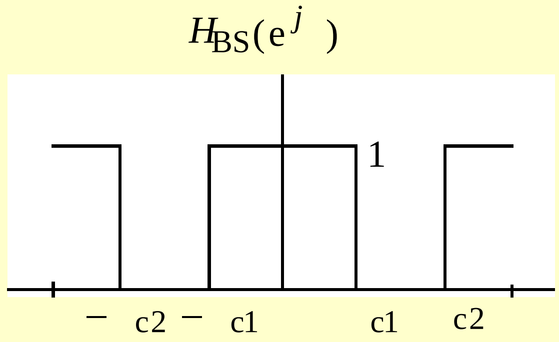
- Ideal bandpass filter -



$$h_{BP}[n] = \begin{cases} \dfrac{\sin(\omega_{c2} n)}{\pi n} - \dfrac{\sin(\omega_{c1} n)}{\pi n}, & n \neq 0 \\ \dfrac{\omega_{c2}}{\pi} - \dfrac{\omega_{c1}}{\pi}, & n = 0 \end{cases}$$

Impulse Responses of Ideal Filters

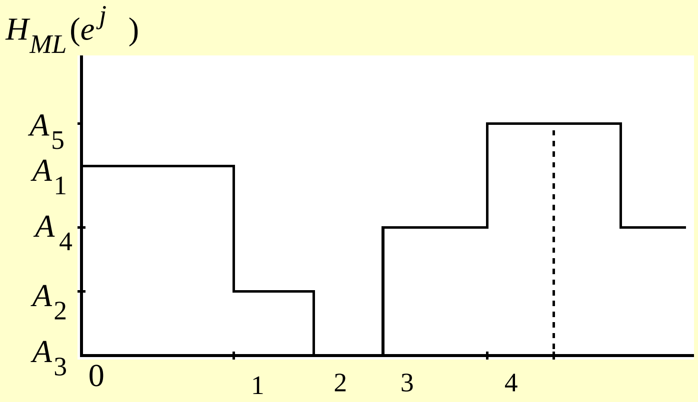
- Ideal bandstop filter -



$$h_{BS}[n] = \begin{cases} 1 - \dfrac{(\omega_{c2} - \omega_{c1})}{\pi}, & n = 0 \\ \dfrac{\sin(\omega_{c1} n)}{\pi n} - \dfrac{\sin(\omega_{c2} n)}{\pi n}, & n \neq 0 \end{cases}$$

Impulse Responses of Ideal Filters

- Ideal multiband filter -



$$H_{ML}(e^{j\omega}) = A_k,$$

$$\omega_{k-1} \le \omega \le \omega_k,$$

$$k = 1, 2, \ldots, L$$

$$h_{ML}[n] = \sum_{\ell=1}^{L} (A_\ell - A_{\ell+1}) \cdot \frac{\sin(\omega_L n)}{\pi n}$$

Impulse Responses of Ideal Filters

- Ideal discrete-time Hilbert transformer -

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0 \\ -j, & 0 < \omega < \pi \end{cases}$$

$$h_{HT}[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 2/\pi \text{n}, & \text{for } n \text{ odd} \end{cases}$$

Impulse Responses of Ideal Filters

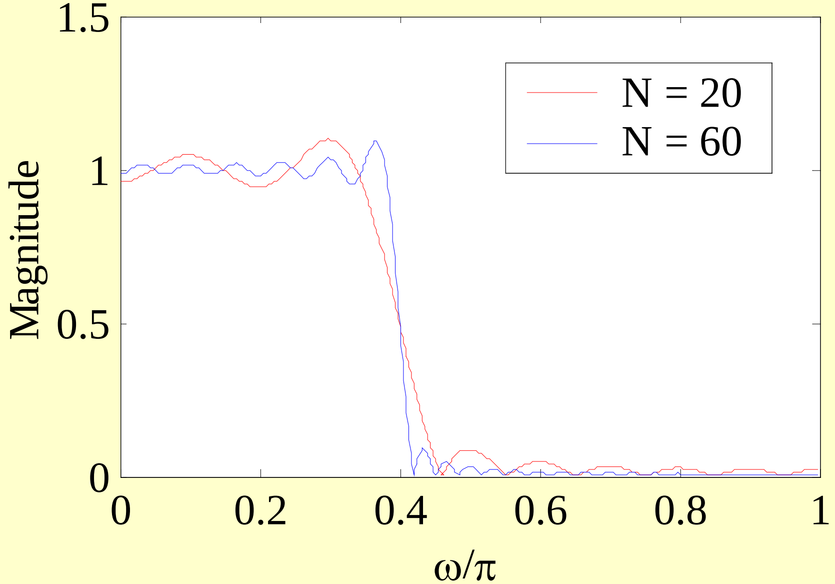
- Ideal discrete-time differentiator -

$$H_{DIF}(e^{j\omega}) = j\omega, \quad 0 \le |\omega| \le \pi$$

$$h_{DIF}[n] = \begin{cases} 0, & n = 0 \\ \dfrac{\cos \pi n}{n}, & n \ne 0 \end{cases}$$

Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Gibbs Phenomenon

- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths
- Height of the largest ripples remain the same independent of length
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters

Gibbs Phenomenon

- Gibbs phenomenon can be explained by treating the truncation operation as an windowing operation:

$$h_t[n] = h_d[n] \cdot w[n]$$

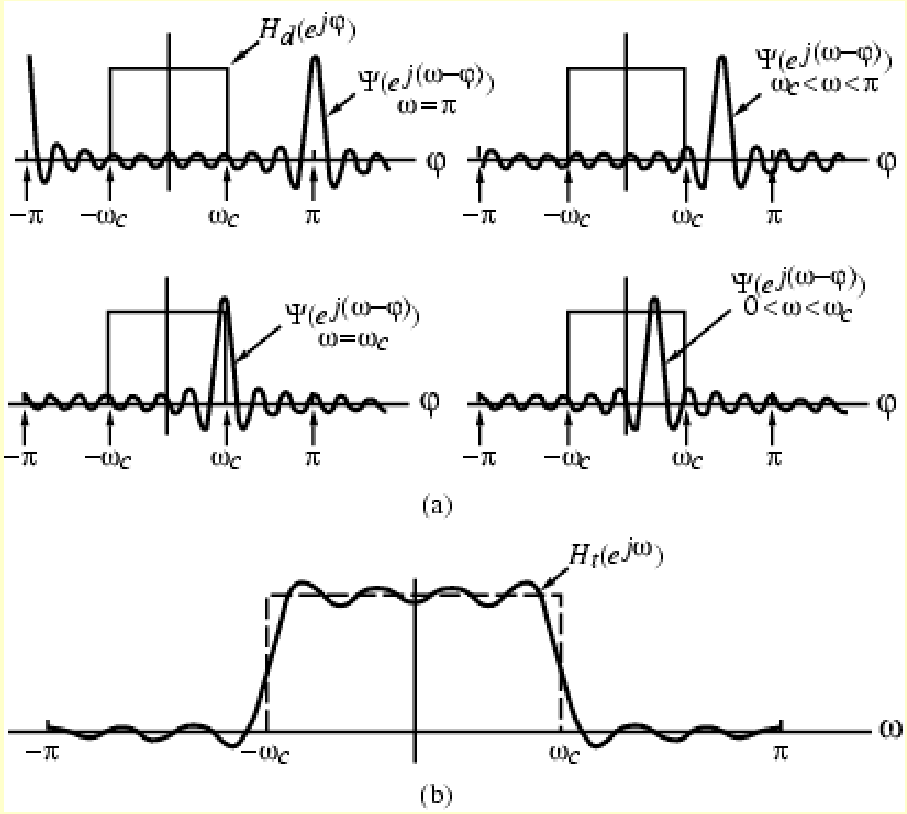
- In the frequency domain

$$H_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\varphi}) \Psi(e^{j(\omega-\varphi)}) \, d\varphi$$

- where $H_t(e^{j\omega})$ and $\Psi(e^{j\omega})$ are the DTFTs of $h_t[n]$ and $w[n]$, respectively

Gibbs Phenomenon

- Thus $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of $H_d(e^{j\omega})$ with $\Psi(e^{j\omega})$



Gibbs Phenomenon

- If $\Psi(e^{j\omega})$ is a very narrow pulse centered at $\omega = 0$ (ideally a delta function) compared to variations in $H_d(e^{j\omega})$, then $H_t(e^{j\omega})$ will approximate $H_d(e^{j\omega})$ very closely
- Length $2M+1$ of $w[n]$ should be very large
- On the other hand, length $2M+1$ of $h_t[n]$ should be as small as possible to reduce computational complexity

Gibbs Phenomenon

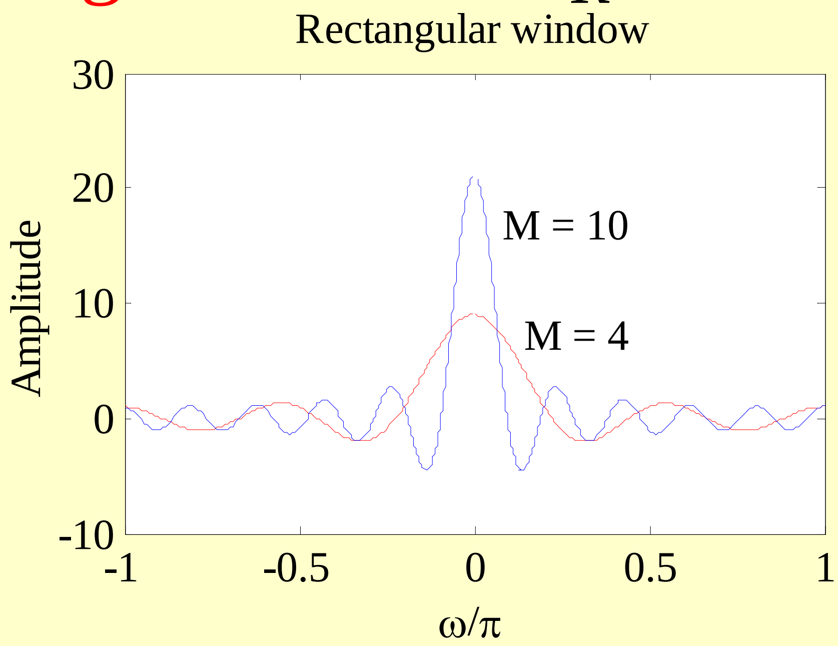
- A **rectangular window** is used to achieve simple truncation:

$$w_R[n] = \begin{cases} 1, & 0 \le |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

- Presence of oscillatory behavior in $H_t(e^{j\omega})$ is basically due to:
  - 1) $h_d[n]$ is infinitely long and not absolutely summable, and hence filter is unstable
  - 2) Rectangular window has an abrupt transition to zero

Gibbs Phenomenon

- Oscillatory behavior can be explained by examining the DTFT $\Psi_R(e^{j\omega})$ of $w_R[n]$:



- $\Psi_R(e^{j\omega})$ has a **main lobe** centered at $\omega = 0$
- Other ripples are called **sidelobes**

Gibbs Phenomenon

- Main lobe of $\Psi_R(e^{j\omega})$ characterized by its width $4\pi/(2M+1)$ defined by first zero crossings on both sides of $\omega = 0$
- As M increases, width of main lobe decreases as desired
- Area under each lobe remains constant while width of each lobe decreases with an increase in M
- Ripples in $H_t(e^{j\omega})$ around the point of discontinuity occur more closely but with no decrease in amplitude as M increases

Gibbs Phenomenon

- Rectangular window has an abrupt transition to zero outside the range $-M \le n \le M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

(1) Using a window that tapers smoothly to zero at each end, or

(2) Providing a smooth transition from passband to stopband in the magnitude specifications