BADJI MOKHTAR ANNABA UNIVERSITY
FACULTY OF TECHNOLOGY
Computer Sciences and Electronics Department
1ST YEAR

Teaching mode : online


KINEMATICS

Planar Motion

Agendas

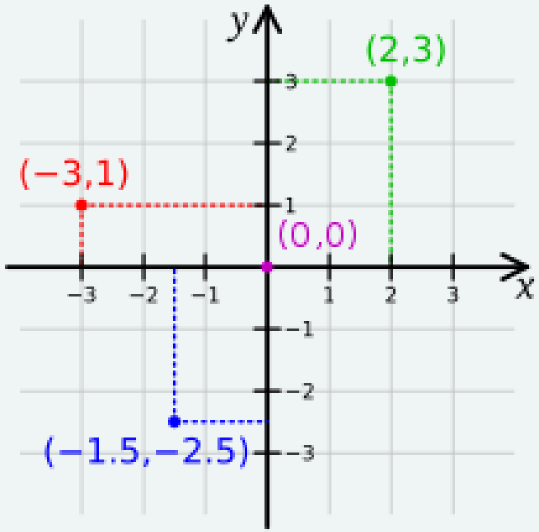
01 Motion in Cartesian Coordinates

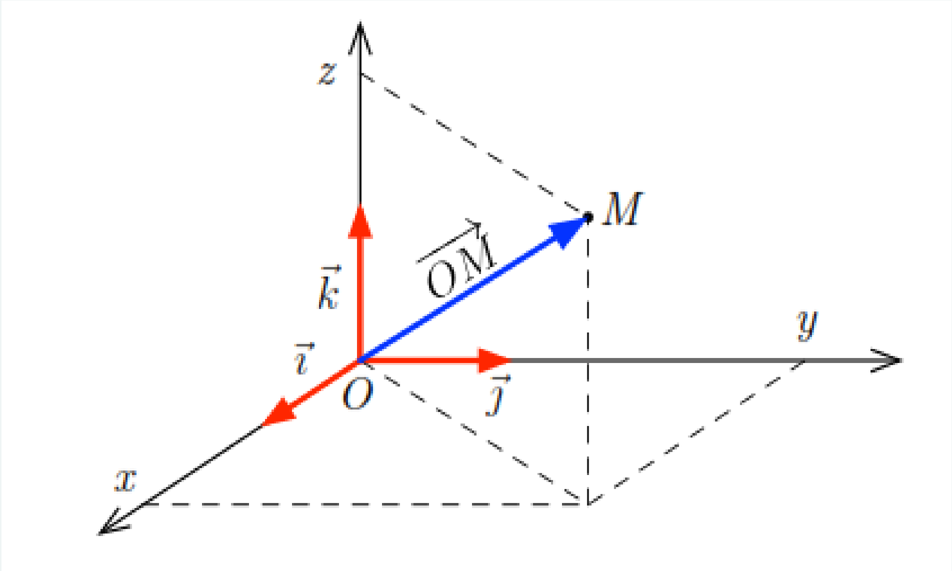
02 Motion in Polar Coordinates

03 Circular motion

04 Frenet frame

Cartesian Coordinates





- The cartesian coordinates of a point M are (x,y,z) The vector $\overrightarrow{OM}$ wish relie between the origin point and point M is writen as :

$$\overrightarrow{OM} = X\vec{i} + Y\vec{j} + Z\vec{k}$$

- **X** : is the axis of the point M.
- **Y**: is the ordonate of the point M.
- **Z**: is the height of the point M.

- **Position vector:**

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

- **Acceleration vector :**

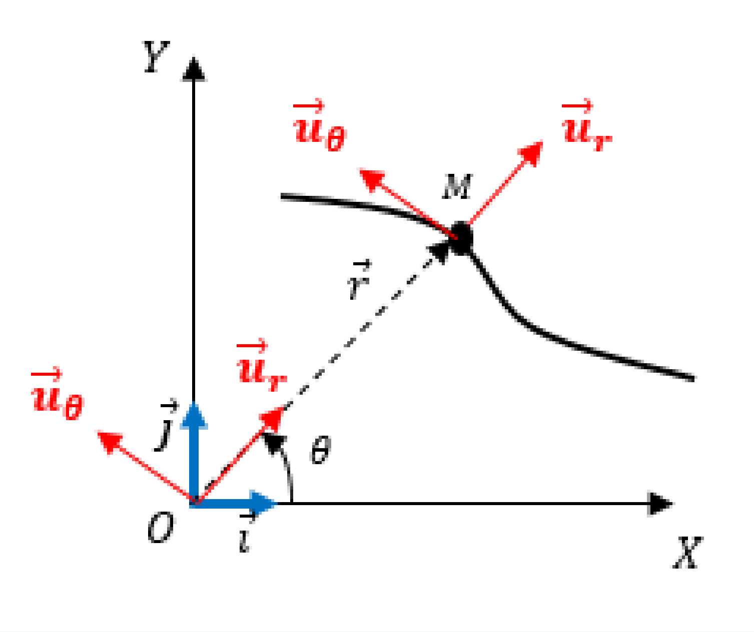
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k}$$

$$= \frac{d^2\overrightarrow{OM}}{dt} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

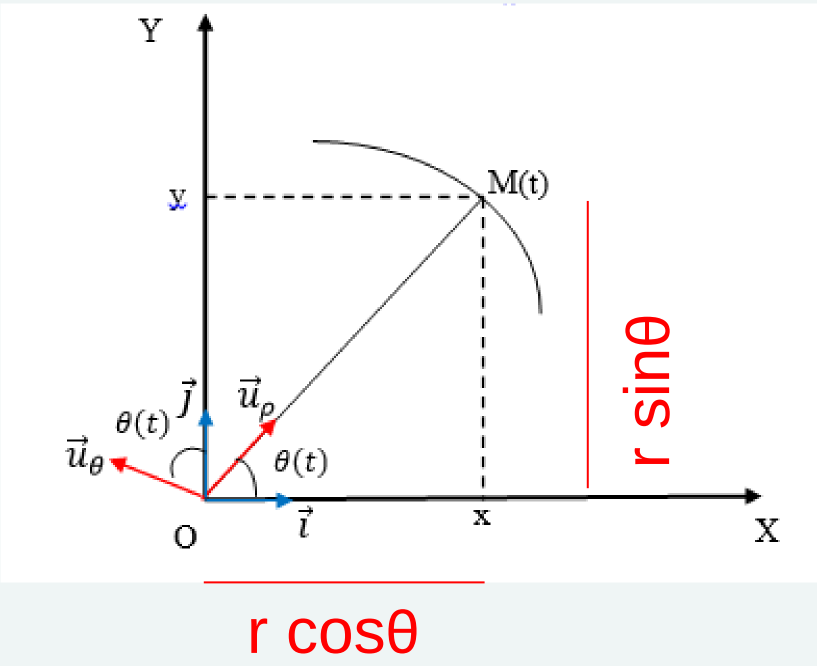
$$\vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

Polar Coordinates

$$0 \leq r < +\infty$$
$$0 \leq \theta < 2\pi$$



$$\|\vec{u_r}\| = \|\vec{u_\theta}\| = 1$$

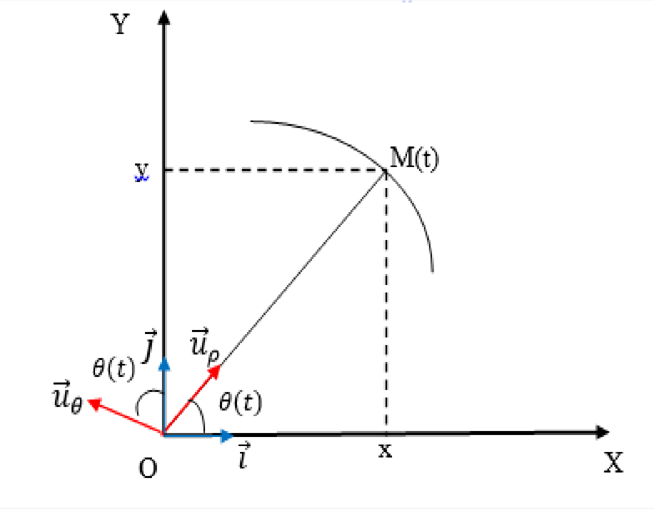


Convert Polar to Cartesian:

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} : \begin{cases} x = r.cos\theta \\ y = r.sin\theta \end{cases}$$

Convert Cartesian to Polar:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} r \\ \theta \end{pmatrix} : \begin{cases} r = \sqrt{x^2 + y^2} \\ \text{tg}(\theta) = \frac{x}{y} \end{cases}$$



- In polar system the coordinates of a point M are (r, θ)

  r: is the distance between the origin and the point M

  θ : is the angle between the axis(x) and the vector OM

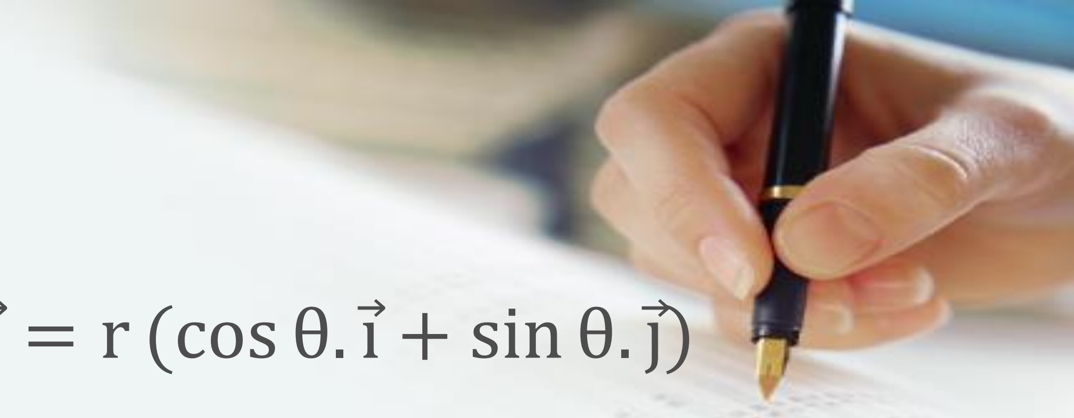
- We define the polar basis by the orthonormal basis

($\vec{Ur}$, $\vec{U\theta}$) such as :

*$\overrightarrow{u_r}$ *et* $\overrightarrow{u_\theta}$: unit vectors of the system

$\overrightarrow{u_r}$ Unit vector following the direction of the position vector $\overrightarrow{OM}$

$\overrightarrow{u_\theta}$: Unit vector perpendicular to $\overrightarrow{u_r}$ by rotating $\pi/2$ of the vector $\overrightarrow{OM}$



In the Cartesian frame:

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} = r\cos\theta.\vec{i} + r\sin\theta.\vec{j} = r\,(\cos\theta.\vec{i} + \sin\theta.\vec{j})$$

where $\vec{u}_r = (\cos\theta.\vec{i} + \sin\theta.\vec{j})$

Position vector: According to polar coordinates, $\overrightarrow{OM}$ is written as follows

$$\overrightarrow{OM}(t) = \left|\overrightarrow{OM}\right|.\overrightarrow{u_r}$$

$$\vec{r} = r.\overrightarrow{u_r}$$

$\vec{u}_r$:The unit vector is mounted on the position vector:

$$\vec{u}_r = \frac{\overrightarrow{OM}}{\left|\overrightarrow{OM}\right|} = \frac{\vec{r}}{r}$$

RQ: θ *and* r depend on time: r = f(t) And θ = g (t)

- The polar base is written in terms of the Cartesian base as follows:

- $$\begin{cases} \overrightarrow{u_r} = cos\theta\,\vec{i} + sin\theta\,\vec{j} \\ \overrightarrow{u_\theta} = -sin\theta\vec{i} + cos\theta\,\vec{j} \end{cases}$$

- $$\begin{cases} \dfrac{d\overrightarrow{u_r}}{dt} = -\dot{\theta}\,sin\theta\,\vec{i} + \dot{\theta}\,cos\theta\,\vec{j} = \dot{\theta}\overrightarrow{u_\theta} \\ \dfrac{d\overrightarrow{u_\theta}}{dt} = -\dot{\theta}\,cos\theta\,\vec{i} - \dot{\theta}\,sin\theta\,\vec{j} = -\dot{\theta}\overrightarrow{u_r} \end{cases}$$

- **Velocity vector**

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}(r.\overrightarrow{u_r}) = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt}$$

$$\boldsymbol{\vec{v}(t) = \frac{dr}{dt}\overrightarrow{u_r} + r\,\frac{d\theta}{dt}\overrightarrow{u_\theta} = \dot{r}\,\overrightarrow{u_r} + r\dot{\theta}\,\overrightarrow{u_\theta}}$$

$$\boldsymbol{\vec{v}(t)} = \boldsymbol{\vec{v}}_r(t) + \boldsymbol{\vec{v}_\theta}(t)$$

Speed has two components

$$v_r(t) = \frac{dr}{dt} = \dot{r} \quad \rightarrow radial \quad (m/s)$$

$$v_\theta(t) = r\,\frac{d\theta}{dt} = r\dot{\theta} \rightarrow transversal \quad (m/s)$$

Magnitude

$$|\vec{v}| = \sqrt{{v_r}^2 + {v_{\boldsymbol{\theta}}}^2} \quad \text{withe} \quad \vec{v} \begin{cases} v_r = \dot{\boldsymbol{r}} \\ v_{\boldsymbol{\theta}} = \boldsymbol{r}\dot{\boldsymbol{\theta}} \end{cases}$$

- Acceleration vector

- $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(\dot{r}\,\overrightarrow{u_r} + r\dot{\theta}\,\overrightarrow{u_\theta}\right) =$
  $\frac{d^2r}{dt^2}\overrightarrow{u_r} + \dot{r}\,\frac{d\overrightarrow{u_r}}{dt} + \frac{dr}{dt}\dot{\theta}\,\overrightarrow{u_\theta} + r\,\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} + r\,\dot{\theta}\,\frac{d\overrightarrow{u_\theta}}{dt}$

- $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \left(\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2\right)\overrightarrow{u_r} + \left(2\,\frac{d\,r}{dt}.\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right)\overrightarrow{u_\theta}$
  $= \left(\ddot{r} - r\dot{\theta}^2\right)\overrightarrow{u_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overrightarrow{u_\theta}$

$$\vec{a}(t) = \vec{a}_r(t) + \vec{a}_\theta(t)$$

$$a_r(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = \ddot{r} - r\dot{\theta}^2 \rightarrow$$

radial component (m/s^2)

$$a_\theta(t) = \left(2\frac{d\,r}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = 2\dot{r}\dot{\theta} + r\ddot{\theta} \rightarrow$$

$orthogonal$ component (m/s^2)

Magnitude

$$|\vec{a}| = \sqrt{{a_r}^2 + {a_\theta}^2} \quad \text{withe } \vec{a}\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 \\ a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{cases}$$

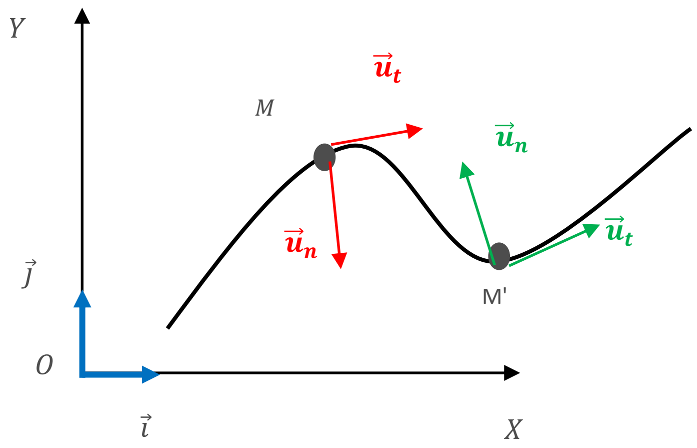
Circular Motion

- In a circular motion, r is constant (r= R)
- So the angular velocity vector has those components
- V = $\begin{cases} v_r = \dot{r} = 0 \\ v_{\theta=} R\dot{\theta} = Rw \end{cases}$
- And the angular acceleration vector has those components
- a =

$$\begin{cases} a_N = R\dot{\theta}^2 = -Rw^2 \quad normal\ acceleration \\ a_{T=} R\ddot{\theta} \quad tengencial\ acceleration \end{cases}$$

Frenet frame (Intrinsic coordinates)

In the case of planar motion, we can define at each point M on the trajectory the Frenet frame. For this purpose, at every point M, we define a vector $\overrightarrow{u_t}$ tangent to the trajectory and oriented in the direction of motion, and we define the $\overrightarrow{u_n}$ perpendicular to $\overrightarrow{u_t}$ and oriented towards the concavity of the trajectory.



$$\vec{v} = v\,\overrightarrow{U_t} = \frac{dS}{dt}\overrightarrow{U_t}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\overrightarrow{U_t} + v\,\frac{d\overrightarrow{U_t}}{dt}$$

$$\vec{a} = \frac{dv}{dt}\overrightarrow{U_t} + v\,\dot{\theta}\overrightarrow{U_n}$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{dS}{dt}\,\frac{1}{R} = \frac{v}{R}$$

$$\vec{a} = \frac{dv}{dt}\overrightarrow{U_t} + \frac{v^2}{R}\overrightarrow{U_n} = \overrightarrow{a_t} + \overrightarrow{a_n}$$

$$\overrightarrow{a_t} = \frac{dv}{dt}\overrightarrow{U_t} \quad ; \quad \overrightarrow{a_n} = \frac{v^2}{R}\overrightarrow{U_n}$$

- The acceleration vector can be decomposed

into a tangential component, called the *tangential acceleration*:

$$\overrightarrow{a_t} = \frac{dv}{dt}\overrightarrow{U_t}$$

- and a normal component called the *normal acceleration*:

$$\overrightarrow{a_n} = \frac{v^2}{R}\overrightarrow{U_n}$$

- $\vec{a} = a_T\vec{u}_T + a_N\vec{u}_N$

- $$a^2 = a_T^2 + a_N^2$$
- We can observe that the magnitude of the normal acceleration component is always positive, indicating that the normal acceleration is always directed towards the concavity of the trajectory.

- If $|\vec{v}|$=cst so $\overrightarrow{a_t}$ tangentiel equal 0 →curvilignar uniform motion :

$$\vec{a} = \vec{a}_n = \frac{v^2}{R}\overrightarrow{u_n}$$

- R radius of the curved path

$$\vec{a} \times \vec{v} = \left(\frac{dv}{dt}\overrightarrow{U_t} + \frac{v^2}{R}\overrightarrow{U_n}\right) \times v\,\overrightarrow{U_t}$$

$$\vec{a} \times \vec{v} = \frac{v^3}{R}(\overrightarrow{U_n} \times \overrightarrow{U_t})$$

$$|\vec{a} \times \vec{v}| = \frac{v^3}{R}$$

$$R = \frac{v^3}{|\vec{a} \times \vec{v}|}$$