Definition 2.9 *We say that f is negligible in front of g in x_0,if $\exists h(x)$ defined in a neighborhood of x_0 such that : $f(x) = h(x).g(x)$ and $\lim\limits_{x\to x_0} h(x) = 0$ and we note in this case*

$$f \underset{x\to x_0}{=} o(g) \Leftrightarrow \forall \varepsilon > 0; \forall x \in v(x_0) : |x - x_0| < \varepsilon \Rightarrow |f(x)| \leq \varepsilon \ |g(x)|$$

$$\Leftrightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0 \quad \text{if } g(x) \neq 0$$

Definition 2.10 *We say that f is domained by g in the neighborhood of x_0 if $\exists b(x)$ defined and bounded (in the neighborhood of x_0) such that $f(x) = b(x).g(x)$,and we note in this case*

$$f \underset{x\to x_0}{=} O(g) \Leftrightarrow \exists M > 0 \text{ et } \exists \delta > 0, \forall x \in v(x_0) \,(\text{neighborhood of } x_0) \Rightarrow |f(x)| \leq M\, |g(x)|$$

that is to say $\frac{f}{g}$ is bounded in the neighborhood of x_0

Definition 2.11 *f and g are equivalent to $v(x_0)$. If $\exists \hbar(x)$ defined at $v(x_0)$ such that $f(x) = \hbar(x).g(x)$ and $\lim\limits_{x\to x_0} \hbar(x) = 1$,and we note in this case*

$$f \sim g \Leftrightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1, \text{ if } g(x) \neq 0$$

Theorem 2.5 • *If $f_1 \sim g_1$ and $f_2 \sim g_2$ then :*

$$f_1 f_2 \sim g_1 g_2 \qquad |f_1| \sim |g_1|$$
$$\frac{f_1}{f_2} \sim \frac{g_1}{g_2} \qquad f_1^{\alpha} \sim g_1^{\alpha}$$

Standard equivalence list

1 • $\sin x \underset{0}{\sim} x$

2 • $\tan x \underset{0}{\sim} x$

3 • $\cos x - 1 \underset{0}{\sim} -\dfrac{x^2}{2}$

4 • $(1-x)^{\alpha} \underset{0}{\sim} 1 - \alpha x$

5 • $(1+x)^{\alpha} \underset{0}{\sim} 1 + \alpha x$

6 • $e^x - 1 \underset{0}{\sim} x$

7 • $\arcsin x \underset{0}{\sim} x$

8 • $\arctan x \underset{0}{\sim} x$

9 • $\cot gx \underset{0}{\sim} \dfrac{1}{x}$

10 • $\ln(1+x) \underset{0}{\sim} x$

Example 2.7 *Calculate the following limits :* $\lim\limits_{x\to 0} \dfrac{\sin x \ln(1+x^2)}{x \tan x}$

we have $\sin x \underset{0}{\sim} x$, $\tan x \underset{0}{\sim} x$ *and* $\ln(1+x^2) \underset{0}{\sim} x^2$

then $\lim\limits_{x\to 0} \dfrac{\sin x \ln(1+x^2)}{x \tan x} \underset{0}{\sim} \lim\limits_{x\to 0} \dfrac{x \ln(1+x^2)}{x.x} \underset{0}{\sim} \lim\limits_{x\to 0} \dfrac{x.x^2}{x.x} = \lim\limits_{x\to 0} x = 0$ *hence* $\lim\limits_{x\to 0} \dfrac{\sin x \ln(1+x^2)}{x \tan x} = 0$

Remark 2.1 *To eliminate the indeterminate form 1^{∞}, we use the form of the following limit*

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e = \lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x$$

$$\lim_{x\to 0} (1-x)^{\frac{1}{x}} = e^{-1} = \lim_{x\to +\infty} \left(1 - \frac{1}{x}\right)^x$$

more general $$\lim_{f(x)\to 0} (1+f(x))^{\frac{1}{f(x)}} = e = \lim_{f(x)\to +\infty} \left(1 + \frac{1}{f(x)}\right)^{f(x)}$$

3 Continuity :

3.1 Continuous function at one point :

Definition 3.1 *Let* f: $D_f \to \mathbb{R}$ *be a function and* $x_0 \in D_f$, *we say that* f *is*

• continuous at point $x_0 \in D_f$ if and only if : $f'(x_0)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$ which is equivalent to

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

• continues to the right of point x_0,iff $\lim_{x \overset{>}{\to} x_0} f(x) = f(x_0) \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D :$ $0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

• continues to the left of point x_0,iff $\lim_{x \overset{<}{\to} x_0} f(x) = f(x_0) \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D :$ $0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

Important : $\lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow \lim_{x \overset{>}{\to} x_0} f(x) = \lim_{x \overset{<}{\to} x_0} f(x) = f(x_0)$

Definition 3.2 *(Sequential Caracterization) :Let $f : D_f \to \mathbb{R}$ be a function and $x_0 \in D_f$. we say that f is continuous at x_0 if and only if for every sequence $(u_n)_n$ of points of D_f such that $\lim_{n \to +\infty} u_n = x_0$,we have $\lim_{n \to +\infty} f(u_n) = f(x_0)$.*

Operations on continuous functions

Proposition 3.1 *Let f and g two maps of D in $\mathbb{R}$ and $x_0 \in D$. If f and g are continuous at x_0 then $f + g$, $f.g$ are continuous at x_0.If $\forall x_0, g(x) \neq 0$,then $\frac{f}{g}$ is continuous at x_0.*

Proposition 3.2 *Let f and g be two maps of D in $\mathbb{R}$ and $x_0 \in D$. If f is continuous at x_0 and g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0*

Remark 3.1 *f is said discontinuous at x_0 if*

• *f is not defined at x_0*
• *the limit exists but different from $f(x)$*
• *the limit does not exist.*

3.2 Continuous Functions over an interval :

Definition 3.3 *We say that the real function* $D_f \to \mathbb{R}$ *is continuous over the entire interval* D_f *if and only if, it is continuous at every point of this interval . that's to say*

$$\forall x_0 \in D_f, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Theorem 3.1 *Let* $[a, b]$ *be a closed and bounded interval and* $f :[a, b] \to \mathbb{R}$ *a continuous map. Then* f *is bounded and reaches its upper and lower bounds.*

$$\exists \alpha \in [a, b] : f(\alpha) = \sup f(x) \ ; x \in [a, b]$$

$$\exists \beta \in [a, b] : f(\beta) = \inf f(x) ; \ x \in [a, b]$$

$$\forall x \in [a, b], \text{ we have } f(\beta) \leq f(x) \leq f(\alpha)$$

Theorem 3.2 ***(intermediate Value Theorem) : If*** f *is a continuous function on a closed interval* $[a, b]$ *such that :*$f(a)$ *and* $f(b)$ *have different signs* $(f(a).f(b) < 0)$ *then* $\exists c \in]a, b[: f(c) = 0$

Theorem 3.3 *Second theorem of intermediate values : Let* f *be a continuous function on an inteval* $[a, b]$ *,then for everything* y *strictly included between* $f(a)$ *and* $f(b)$ *we have* $\exists c \in [a, b]$ *such that* $f(c) = y$

Corollary 3.1 *1) The image of any interval* $I \subset \mathbb{R}$, *of a continuous function on* I *will necessarily be an interval.*

2) The 'image of any closed interval $[a, b]$ *,of a continuous function on* $[a, b]$ *will necessarily be a closed interval.*

Application : Let $f :[0, 1] \to [0, 1]$ continue
Show that f admits at least one fixed point i.e $\exists x \in [0, 1]$ such that $f(x) = x$
let $g(x) = f(x) - x$, it is clear that g is continuous on $[0, 1]$
we have $g(0) = f(0) - 0 = f(0) \geq 0 \quad ; 0 \leq f(0) \leq 1$

$g(1) = f(1) - 1 \leq 0 \quad ; \quad 0 \leq f(1) \leq 1$

So according to the intermediate value theorem $\exists x \in [0,1] : g(x) = 0 \Leftrightarrow f(x) - x = 0$.
Hence $\exists x \in [0,1] : f(x) = x$

3.3 Uniforme continuity :

Definition 3.4 *We say that the real function $f : D \to \mathbb{R}$ is uniformly continuous on the interval D if only if, it satisfies*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D; x_1 < x_2 : |x_2 - x_1| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon$$

Remark 3.2 1› *In uniform continuity over an interval the constant δ depends only on the choice of ε, on the other hand in simple continuity over an interval, δ depends on the choice of x_0 and ε*

2› *Uniform continuity on an interval ■ continuity on an interval, it is enough to take $x_2 = x$ and $x_1 = x_0$ in the definition of continuity*

Theorem 3.4 ***(Heine) :*** *Any continuous function on $[a, b]$ is uniformly continuous on this interval.*

Example 3.1 • *The function $f(x) = \alpha x, \alpha \neq 0$ is continuous on $\mathbb{R}$. Indeed*

$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D; x_1 < x_2 : |x_2 - x_1| < \delta \Rightarrow |\alpha x_2 - \alpha x_1| = |\alpha| |x_2 - x_1| < \varepsilon$

we take $\delta = \dfrac{\varepsilon}{|\alpha|}$

• on the other hand the function $f(x) = \dfrac{1}{x}$ is not uniformly continuous on $]0, 1]$ because

$\exists \varepsilon = 1, \forall \delta > 0, \exists x_1 = \dfrac{1}{2n}, n \in \mathbb{N}^*, \exists x_2 = \dfrac{1}{n} : |x_2 - x_1| = \left| \dfrac{1}{n} - \dfrac{1}{2n} \right| = \dfrac{1}{n} < \delta$ and $|f(x_2) - f(x_1)| = \left| \dfrac{1}{x_2} - \dfrac{1}{x_1} \right| = n \geq 1$.

3.4 Monotonic functions et la continuité :

Theorem 3.5 *Let $f : I \to \mathbb{R}$ be a function defined and continuous on the interval I.Then*

$$f \text{ is continuous on } I \Leftrightarrow f(I) \text{ is an interval}$$

Theorem 3.6 *Let $f : I \to \mathbb{R}$ be a function defined and continuous on the interval I.If f is strictly monotonic on I, then f is injective on $J = f(I)$*

Theorem 3.7 *Let $f : I \to \mathbb{R}$ be a function. $(I \subset \mathbb{R}$ is an interval$)$.*

If f is strictly monotonic and continuous on I, then

1. f is a bijection of I in $J = f(I)$

2. The inverse function $f^{-1} : J = f(I) \to I$ is strictly monotonic and continuous onr J

Exemple : Arcsinus function is a reciprocal function of the sine function on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$:

$$\begin{aligned} \arcsin :\ [-1,1] &\to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ x &\mapsto \arcsin x, \end{aligned}$$

and checks the following relationships :

›If $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$: $\sin x = y \Leftrightarrow x = \arcsin y$

›$\sin(\arcsin x) = x, \quad \forall x \in [-1,1],$

›$\arcsin(\sin x) = x, \quad \forall x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right],$

›$\cos(\arcsin x) = \sqrt{1-x^2}, \ \forall x \in [-1,1],$

3.5 Extension by continuity :

Definition 3.5 *Let I be an interval, x_0 a point of I. If the function is not defined at the point $x_0 \in I$ and it admits at this point a finite limit denoted l, the function defined by :*

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ l & \text{if } x = x_0 \end{cases}$$

is called an extension by continuity of f at the point x_0.

Example 3.2 *The function $f(x) = x \sin \frac{1}{x}$ is defined and continues on $\mathbb{R}^*$, and for all $x \in \mathbb{R}^*$ we have*

$|f(x)| = \left| x \sin \frac{1}{x} \right| \leq |x|$ *therefore* $\lim_{x \to 0} f(x) = 0$. *the extension by continuity of f at point $x_0 = 0$ is therefore the function $\widetilde{f}$ defined by :*

$$\widetilde{f}(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Example 3.3 *Find an extension by continuity at $\mathbb{R}$ of the following function :*

$$f(x) = \frac{x^3 + 5x + 6}{x^3 + 1}, \qquad D_f = \mathbb{R} - \{-1\}$$

We have $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^3 + 5x + 6}{x^3 + 1} = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 6)}{(x+1)(x^2 - x + 1)} = \lim_{x \to -1} \frac{x^2 - x + 6}{x^2 - x + 1} = \frac{8}{3}$

The extension $\breve{f}$ of f defined by

$$\widetilde{f}(x) = \begin{cases} \frac{x^3 + 5x + 6}{x^3 + 1} & \text{if } x \neq -1 \\ \frac{8}{3} & \text{if } x = -1 \end{cases}$$